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Inequalities for Generalized Hypergeometric Functions of Two Variables*

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In a previous paper, we developed lower and upper bounds for the generalized hypergeometric functions ${}_pF_q$, $p = q$, $p = q + 1$, and certain confluent forms under appropriate restrictions on the variable and parameters. In the present paper, we extend these notions and obtain similar inequalities for certain generalized hypergeometric functions of two variables.

1. INTRODUCTION

In a previous study [1], we uncovered two sided inequalities for the generalized hypergeometric functions ${}_pF_q(\alpha_p; \rho_q; -z)$ with $p = q, p = q + 1$, and certain confluent forms under rather liberal conditions on the parameters and variable. In this paper, we extend our ideas to get similar inequalities for certain hypergeometric functions of two variables. To this end, we make free use of the notation and pertinent theorems in [1]. We also assume that the reader is sufficiently informed of standard results on hypergeometric functions of two variables given in [2] and [3].

As in the case of ${}_2F_1(z)$, hypergeometric functions of two variables of order two associated with the names of Appell and Horn can be basically represented by Eulerian integrals. The simple representations are by means of double integrals. In each case, there are also representations by means of a single integral. Here, in most cases, the integrand contains a hypergeometric function of a single variable or possibly a product of such functions.

There are 34 distinct convergent series of order two. Of these, we shall deal in some detail only with the Appell functions F_1, F_2 and F_3. The F_4 function is briefly noted to illustrate a difficulty in getting simple inequalities. Our treatment is sufficient to illustrate the techniques involved and the kinds of results expected.

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2. Inequalities for F_1

We start with the integral representation

$$F_1 \equiv F_1(\alpha, \beta, \beta' \gamma; -x, -y)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta-\beta')} \int_0^1 \int_0^1 s^{\beta+\beta'-1}(1-s)^{\gamma-\beta-\beta'-1}\, t^{\beta'-1}(1-t)^{\beta-1}$$

$$\times\ [1 + xs(1-t) + sty]^{-\alpha}\, ds\, dt,$$

$$R(\gamma) > R(\beta+\beta'), \qquad R(\beta) > 0, \qquad R(\beta') > 0,$$

$$|\arg(1+x)| < \pi, \qquad |\arg(1+y)| < \pi. \tag{1}$$

Expand $[1 + xs(1-t) + sty]^{-\alpha}$ by the binomial theorem and apply known results for beta integrals. Then we get the double series

$$F_1 = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n(-x)^m(-y)^n}{(\gamma)_{m+n}\, m!\,n!},$$

$$|x| < 1, \qquad |y| < 1, \tag{2}$$

where we naturally suppose that γ is not a negative integer or zero. It is useful to note that

$$F_1(\alpha, \beta, \beta', \gamma; -x, -y) = F_1(\alpha, \beta', \beta, \gamma; -y, -x). \tag{3}$$

Apply the inequality (4.4) of [1] to $[1 + xs(1-t) + sty]^{-\alpha}$ in (1) and use (1.6) of [1]. Then

$$\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\,\Gamma(\beta')} \int_0^1 t^{\beta'-1}(1-t)^{\beta-1}{}_2F_1\left(\begin{matrix}1, \beta+\beta' \\ \gamma\end{matrix} \,\middle|\, -\alpha\{x(1-t)+yt\}\right) dt$$

$$< F_1 < \frac{1-\alpha}{2+\alpha} + \frac{2\alpha\Gamma(\beta+\beta')}{(1+\alpha)\,\Gamma(\beta)\,\Gamma(\beta')} \int_0^1 t^{\beta'-1}(1-t)^{\beta-1}$$

$$\times\ {}_2F_1\left(\begin{matrix}1, \beta+\beta' \\ \gamma\end{matrix} \,\middle|\, -\frac{(1+\alpha)}{2}\{x(1-t)+yt\}\right) dt,$$

$$0 \leqslant \alpha \leqslant 1, \qquad \gamma > \beta+\beta', \quad \beta > 0, \quad \beta' > 0, \quad x > 0, \quad y > 0. \tag{4}$$

Next, apply (4.6) of [1] appropriately to the ${}_2F_1$'s in (4) and again use (1.6) of [1]. We find the following theorem.

THEOREM 1.

$$\left[1 + \frac{\alpha}{\gamma}(\beta x + \beta' y)\right]^{-1} < F_1 < \frac{1-\alpha}{1+\alpha} + \frac{2\alpha}{1+\alpha}\left[\frac{(\gamma - \beta - \beta')}{\gamma(\beta + \beta' + 1)}\right.$$
$$+ \frac{(\gamma + 1)}{\gamma(\beta + \beta' + 1)(\beta' + 1)}\left\{\frac{\beta}{1 + ((1+\alpha)/2(\gamma+1))(\beta + \beta' + 1)\,x}\right.$$
$$\left.\left. + \frac{\beta'(\beta + \beta' + 1)}{1 + ((1+\alpha)/2(\gamma+1))[\beta x + (\beta' + 1)\,y]}\right\}\right],$$
$$0 \leqslant \alpha \leqslant 1, \quad \gamma > \beta + \beta', \quad \beta > 0, \quad \beta' > 0, \quad y \geqslant x > 0. \tag{5}$$

Equation (3) states that F_1 is unchanged if $\beta \leftrightarrow \beta'$ and $x \leftrightarrow y$ simultaneously. The left inequality of (5) possesses this symmetry feature, but this is not so for the right side. If we have to do with $x \geqslant y > 0$, then in the right hand inequality of (5), replace β and x by β' and y, respectively. If $x = y$, $F_1 = {}_2F_1(1, \beta + \beta'; \gamma; -x)$ and (5) reduces to (4.7) of [1].

Now F_1 has a representation in terms of a single integral from which we can deduce another inequality. We have

$$F_1 = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma - \alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1+ux)^{-\beta}(1+uy)^{-\beta'}\,du,$$
$$R(\gamma) > R(\alpha) > 0, \qquad |\arg(1+x)| < \pi, \qquad |\arg(1+y)| < \pi. \tag{6}$$

Apply (4.4) of [1] to $(1 + ux)^{-\beta}$ and $(1 + uy)^{-\beta'}$. When the inequalities are multiplied, terms of the form

$$\frac{1}{(1+au)(1+bu)} = \frac{1}{(a-b)}\left[\frac{a}{1+au} - \frac{b}{1+bu}\right]$$

appear with $a > b > 0$. Then with the aid of (1.6) of (1), we get

$$\left(\frac{\beta' y}{\beta' y - \beta x}\right){}_2F_1\left(\begin{matrix}1, \alpha\\ \gamma\end{matrix}\,\middle|\, -\beta' y\right) - \left(\frac{\beta x}{\beta' y - \beta x}\right){}_2F_1\left(\begin{matrix}1, \alpha\\ \gamma\end{matrix}\,\middle|\, -\beta x\right)$$
$$< F_1 < \left(\frac{1-\beta}{1+\beta}\right)\left(\frac{1-\beta'}{1+\beta'}\right) + \left(\frac{1-\beta'}{1+\beta'}\right)\left(\frac{2\beta}{1+\beta}\right){}_2F_1\left(\begin{matrix}1, \alpha\\ \gamma\end{matrix}\,\middle|\, -\left(\frac{1+\beta}{2}\right)x\right)$$
$$+ \left(\frac{1-\beta}{1+\beta}\right)\left(\frac{2\beta'}{1+\beta'}\right){}_2F_1\left(\begin{matrix}1, \alpha\\ \gamma\end{matrix}\,\middle|\, -\left(\frac{1+\beta'}{2}\right)y\right)$$
$$+ \frac{4\beta\beta' y}{(1+\beta)[(1+\beta')\,y - (1+\beta)\,x]}\,{}_2F_1\left(\begin{matrix}1, \alpha\\ \gamma\end{matrix}\,\middle|\, -\left(\frac{1+\beta'}{2}\right)y\right)$$
$$- \frac{4\beta\beta' x}{(1+\beta')[(1+\beta')\,y - (1+\beta)x)]}\,{}_2F_1\left(\begin{matrix}1, \alpha\\ \gamma\end{matrix}\,\middle|\, -\left(\frac{1+\beta}{2}\right)x\right),$$
$$0 \leqslant \beta \leqslant 1, \quad 0 \leqslant \beta' \leqslant 1, \quad \gamma > \alpha > 0, \quad \beta' \geqslant \beta, \quad y \geqslant x > 0. \tag{7}$$

To each ${}_2F_1$ in (7) apply (4.6) of [1] appropriately, and so obtain the next theorem.

THEOREM 2.

$$\left(\frac{\beta' y}{\beta' y-\beta x}\right)\left(1+\frac{\alpha\beta' y}{\gamma}\right)^{-1}-\left(\frac{\beta x}{\beta' y-\beta x}\right)$$
$$\times\left\{\frac{\gamma-\alpha}{\gamma(\alpha+1)}+\frac{\alpha(\gamma+1)}{(\alpha+1)\gamma}\left[1+\frac{(\alpha+1)\beta x}{\gamma+1}\right]^{-1}\right\}$$
$$< F_1 < \left(\frac{1-\beta}{1+\beta}\right)\left(\frac{1-\beta'}{1+\beta'}\right)+\frac{(\gamma-\alpha)}{\gamma(\alpha+1)}$$
$$\times\left\{\left(\frac{1-\beta}{1+\beta}\right)\left(\frac{2\beta'}{1+\beta'}\right)+\left(\frac{1-\beta'}{1+\beta'}\right)\left(\frac{2\beta}{1+\beta}\right)\right.$$
$$\left.+\frac{4\beta\beta' y}{(1+\beta)[(1+\beta')y-(1+\beta)x]}\right\}$$
$$+\frac{\alpha(\gamma+1)}{\gamma(\alpha+1)}\left\{\frac{((1-\beta')/(1+\beta'))(2\beta/(1+\beta))}{1+((\alpha+1)/(\gamma+1))((1+\beta/2)x}\right.$$
$$+\frac{((1-\beta)/(1+\beta))(2\beta'/(1+\beta'))}{1+((\alpha+1)/(\gamma+1))((1+\beta')/2)y}$$
$$\left.+\frac{4\beta\beta' y(1+\beta)^{-1}[(1+\beta')y-(1+\beta)x]^{-1}}{1+((\alpha+1)/(\gamma+1))((1+\beta')/2)y}\right\}$$
$$-\frac{4\beta\beta' x(1+\beta')^{-1}[(1+\beta')y-(1+\beta)x]^{-1}}{1+(\alpha/\gamma)((1+\beta)/2)x},$$
$$0\leqslant\alpha\leqslant 1,\quad \gamma\geqslant\alpha\geqslant 0,\quad \gamma>\beta+\beta',\quad \beta>0,\quad \beta'>0,\quad y\geqslant x>0. \tag{8}$$

From (4.13) and (4.16) of [1], we have inequalities for $(1+z)^{-\alpha}$ valid for $1<\alpha<2$ and $-1<\alpha<0$, respectively. These can be used to get further inequalities for F_1 and the other hypergeometric functions of two variables, but we omit such considerations.

3. INEQUALITIES FOR F_2

We have the integral representation

$$F_2\equiv F_2(\alpha,\beta,\beta',\gamma,\gamma';-x,-y)=\frac{\Gamma(\gamma)\,\Gamma(\gamma')}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta)\,\Gamma(\gamma-\beta')}$$
$$\times\int_0^1\int_0^1\frac{u^{\beta-1}(1-u)^{\gamma-\beta-1}\,v^{\beta'-1}(1-v)^{\gamma'-\beta'-1}}{(1+ux+vy)^{\alpha}}\,du\,dv,$$
$$R(\gamma)>R(\beta)>0,\quad R(\gamma')>R(\beta')>0,\quad |\arg(1+x+y)|<\pi. \tag{9}$$

The double power series representation is

$$F_2 = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n(-x)^m(-y)^n}{(\gamma)_m(\gamma')_n\, m!n!}, \qquad |x+y|<1 \tag{10}$$

with the further restriction that neither γ nor γ' is a negative integer or zero. The symmetry relation is

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; -x, -y) = F_2(\alpha, \beta', \beta, \gamma', \gamma; -y, -x). \tag{11}$$

THEOREM 3.

$$\begin{aligned}
\left[1+\frac{\alpha\beta x}{\gamma}+\frac{\alpha\beta' y}{\gamma'}\right]^{-1} < F_2 < \frac{1-\alpha}{1+\alpha} &+ \frac{2\alpha}{(1+\alpha)(\beta+1)(\beta'+1)\gamma\gamma'} \\
\times \Big\{(\gamma-\beta)(\gamma'-\beta') &+ \frac{\beta(\gamma+1)(\gamma'-\beta')}{1+((1+\alpha)/2)((\beta+1)/(\gamma+1))\,x} \\
&+ \frac{\beta'(\gamma'+1)(\gamma-\beta)}{1+((1+\alpha)/2)((\beta'+1)/(\gamma'+1))\,y} \\
&+ \frac{\beta\beta'(\gamma+1)(\gamma'+1)}{1+((1+\alpha)/2)[((\beta+1)/(\gamma+1))\,x+((\beta'+1)/(\gamma'+1))\,y]}\Big\},
\end{aligned}$$

$$\gamma>\beta>0, \qquad \gamma'>\beta'>0, \qquad 0\leqslant\alpha\leqslant 1, \qquad x>0, \qquad y>0. \tag{12}$$

Proof. The procedure is much akin to that for Theorems 1 and 2. Apply (4.4) of [1] to $(1+ux+vy)^{-\alpha}$ in (9) and use (1.6) of [1]. Then under the conditions of the theorem,

$$\begin{aligned}
&\frac{\Gamma(\gamma')}{\Gamma(\beta')\,\Gamma(\gamma'-\beta')}\int_0^1 \frac{v^{\beta'-1}(1-v)^{\gamma'-\beta'-1}}{1+\alpha y v}\,{}_2F_1\left(\begin{matrix}1,\beta\\ \gamma\end{matrix}\,\middle|\,-\frac{\alpha x}{1+\alpha y v}\right)dv \\
&\quad < F_2 < \frac{1-\alpha}{1+\alpha} + \frac{\Gamma(\gamma')}{\Gamma(\beta')\,\Gamma(\gamma'-\beta')}\left(\frac{2\alpha}{1+\alpha}\right)\int_0^1 \frac{v^{\beta'-1}(1-v)^{\gamma'-\beta'-1}}{1+((1+\alpha)/2)\,yv} \\
&\quad \times {}_2F_1\left(\begin{matrix}1,\beta\\ \gamma\end{matrix}\,\middle|\,-\frac{((1+\alpha)/2)\,x}{1+((1+\alpha)/2)\,yv}\right)dv.
\end{aligned} \tag{13}$$

Next apply (4.6) of [1] appropriately to each ${}_2F_1$ in (13). Use (1.6) of [1] to integrate. Each side of the resulting inequality will involve a ${}_2F_1$ one of whose numerator arguments is unity. So (4.6) of [1] is again applicable and we arrive at (12).

The two-sided inequality (12) is symmetric in the sense of (11).

4. Inequalities for F_3

Consider

$$\begin{aligned} F_3 &\equiv F_3(\alpha, \alpha'; \beta, \beta', \gamma, -x, -y) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\rho)\,\Gamma(\rho')} \int_0^1 t^{\rho-1}(1-t)^{\rho'-1} \\ &\quad \times {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \rho\end{matrix} \,\middle|\, -xt\right) {}_2F_1\left(\begin{matrix}\alpha', \beta' \\ \rho'\end{matrix} \,\middle|\, -y(1-t)\right) dt, \\ &\quad \rho + \rho' = \gamma, \qquad R(\rho) > 0, \qquad R(\rho') > 0, \\ &\quad |\arg(1+x)| < \pi, \qquad |\arg(1+y)| < \pi, \end{aligned} \tag{14}$$

$$F_3 = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n(-x)^m(-y)^n}{(\gamma)_{m+n}\, m!n!}, \qquad |x| < 1, \qquad |y| < 1, \tag{15}$$

γ is not a negative integer or zero, and

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; -x, -y) = F_3(\alpha', \alpha, \beta', \beta, \gamma; -y, -x). \tag{16}$$

THEOREM 4.

$$\begin{aligned} \left(1 + \frac{\alpha\beta x}{\gamma}\right)^{-1}\left(1 + \frac{\alpha'\beta' y}{\gamma}\right)^{-1} < F_3 &< \left[1 - \frac{u(\rho+1)}{\rho}\right]\left[1 - \frac{u'(\rho'+1)}{\rho'}\right] \\ &+ \left[1 - \frac{u(\rho+1)}{\rho}\right] \frac{u'}{\rho'\gamma} \left[\rho + \frac{\rho'(\gamma+1)}{1+v'}\right] \\ &+ \left[1 - \frac{u'(\rho'+1)}{\rho'}\right] \frac{u}{\rho\gamma} \left[\rho' + \frac{\rho(\gamma+1)}{1+v}\right] \\ &+ \frac{uu'}{\rho\rho'\gamma[v(\rho'+1) + vv'(\gamma+1) + v'(\rho+1)]} \\ &\times \left[v\rho'(\rho'+1)^2 + \frac{v(\gamma+1)\,\rho(\rho'+1)^2}{1+v} \right. \\ &\left. + v'\rho(\rho+1)^2 + \frac{v'(\gamma+1)\,\rho'(\rho+1)^2}{1+v'}\right], \\ u = \frac{2\alpha\beta}{(\alpha+1)(\beta+1)}, &\qquad u' = \frac{2\alpha'\beta'}{(\alpha'+1)(\beta'+1)}, \\ v = \frac{(\alpha+1)(\beta+1)\,x}{2(\gamma+1)}, &\qquad v' = \frac{(\alpha'+1)(\beta'+1)\,y}{2(\gamma+1)}, \\ \rho + \rho' = \gamma, \qquad 0 \leqslant \alpha \leqslant \rho &\leqslant \gamma - \alpha' \leqslant \gamma, \qquad 0 \leqslant \beta \leqslant 1, \\ 0 \leqslant \beta' \leqslant 1, &\qquad x > 0, \qquad y > 0. \end{aligned} \tag{17}$$

Proof. Apply (4.7) of [1] to each ${}_2F_1$ in (14). In the algebra we encounter the form

$$\{(1+at)[1+b(1-t)]\}^{-1} = (a+b+ab)^{-1}\left[\frac{a}{1+at} + \frac{b}{1+b(1-t)}\right].$$

Then with the aid of (1.6) of [1], we have

$$\begin{aligned}
&(a+b+ab)^{-1}\left[a\,{}_2F_1\left(\begin{matrix}1,\rho\\ \gamma\end{matrix}\,\middle|\,-a\right) + b\,{}_2F_1\left(\begin{matrix}1,\rho'\\ \gamma\end{matrix}\,\middle|\,-b\right)\right] < F_3 \\
&\quad < \left[1 - \frac{(\rho+1)\,u}{\rho}\right]\left[1 - \frac{(\rho'+1)\,u'}{\rho'}\right] \\
&\quad + \left[1 - \frac{(\rho+1)\,u}{\rho}\right]\frac{u'(\rho'+1)}{\rho'}\,{}_2F_1\left(\begin{matrix}1,\rho\\ \gamma\end{matrix}\,\middle|\,-\frac{v'(\gamma+1)}{\rho'+1}\right) \\
&\quad + \left[1 - \frac{(\rho'+1)\,u'}{\rho'}\right]\frac{u(\rho+1)}{\rho}\,{}_2F_1\left(\begin{matrix}1,\rho\\ \gamma\end{matrix}\,\middle|\,-\frac{v(\gamma+1)}{\rho+1}\right) \\
&\quad + \frac{uu'(\rho+1)(\rho'+1)}{\begin{pmatrix}\rho\rho'[(v(\gamma+1)/(\rho+1)) + (vv'(\gamma+1)^2/(\rho+1)(\rho'+1)) \\ + (v'(\gamma+1)/(\rho'+1))]\end{pmatrix}} \\
&\quad \times \left\{\frac{v(\gamma+1)}{\rho+1}\,{}_2F_1\left(\begin{matrix}1,\rho\\ \gamma\end{matrix}\,\middle|\,-\frac{v(\gamma+1)}{\rho+1}\right) + \frac{v'(\gamma+1)}{\rho'+1}\right. \\
&\quad \left.\times\,{}_2F_1\left(\begin{matrix}1,\rho'\\ \gamma\end{matrix}\,\middle|\,-\frac{v'(\gamma+1)}{\rho'+1}\right)\right\}, \\
&\qquad\qquad a = \frac{\alpha\beta x}{\rho}, \qquad b = \frac{\alpha'\beta' y}{\rho'}, \\
&\rho+\rho' = \gamma, \qquad 0 \leqslant \alpha \leqslant \rho \leqslant \gamma - \alpha' \leqslant \gamma, \qquad 0 \leqslant \beta \leqslant 1, \\
&\qquad 0 \leqslant \beta' \leqslant 1, \qquad x > 0, \qquad y > 0.
\end{aligned} \tag{18}$$

Now use (4.6) of [1] to get the stated result.

For convenience in the ensuing discussion let $M(\rho)$ designate the right side of (17). Both F_3 and the left side of (17) are independent of ρ. This is not so for $M(\rho)$, as can be verified by a numerical example which is presented later. In this connection, neither the left side nor the right side of (18) is independent of ρ. In practice one should use that value of ρ for which $M(\rho)$ is a minimum. Here ρ, of course, must be restricted by the inequality conditions given in (17). We now show how this can be readily accomplished. First we observe that $\rho\rho' M(\rho)$ vanishes when $\rho = 0$ and when $\rho = \gamma$. This implies that $M(\rho)$ has the form

$$M(\rho) = \frac{a\rho + b}{c\rho + d}. \tag{19}$$

Thus three independent conditions are sufficient to determine this linear fraction. Clearly $M(\rho)$ has no relative extremal points. By use of L'Hospital's theorem, we have

$$\begin{aligned}\gamma M(0) &= (1-u)[\gamma - u'(\gamma+1)] + \frac{uu'(\gamma - v')}{\gamma(1+v')} + \frac{u'(1-u)(\gamma+1)}{1+v'} \\ &+ \frac{u[\gamma - u'(\gamma+1)](\gamma - v)}{\gamma(1+v)} + \frac{uu'}{\gamma[v(\gamma+1) + vv'(\gamma+1) + v]} \\ &\times \Big[v' - v(\gamma+1)(3\gamma+1) + \frac{v(\gamma+1)^3}{1+v} + \frac{(\gamma+1)}{1+v'} \\ &\times \{v'(\gamma-1) + v\gamma\}\Big], \end{aligned} \tag{20}$$

and $\gamma M(\gamma)$ is given by the right side of (20) when u and v are interchanged with u' and v', respectively. For a third condition, let $\rho \to \infty$. Since γ is fixed, we take $\rho/\rho' \to -1$. Then

$$\begin{aligned}\gamma M(\infty) &= \gamma(1-u)(1-u') - (u + u' - uu') + (\gamma+1) \\ &\times \left[\frac{(1-u)\,u'}{1+v'} + \frac{(1-u')\,u}{1+v} + \frac{uu'}{(1+v)(1+v')}\right]. \end{aligned} \tag{21}$$

Of course, further conditions useful for check purposes in numerical work can be obtained by evaluating $M(\rho)$ for specific values of ρ. $M(\rho)$ follows by solution of the equations

$$\frac{b}{d} = M(0), \qquad \frac{a\gamma + b}{c\gamma + d} = M(\gamma), \qquad \frac{a}{c} = M(\infty). \tag{22}$$

Since $M(\infty)$ is finite, we can take $c = 1$. So

$$a = M(\infty), \qquad b = dM(0), \qquad d = \frac{\gamma[R(\gamma) - R(\infty)]}{R(0) - R(\gamma)}, \tag{23}$$

and it is a simple matter to find that ρ which makes $M(\rho)$ a minimum subject to the conditions specified in (17).

We can get a further inequality by starting with another integral representation of F_3 which can be derived from (14). There put $\rho = \beta$ whence $\rho' = \gamma - \beta$ and replace t by $1 - t$ in the integrand. Thus

$$F_3 = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^1 \frac{t^{\gamma-\beta-1}(1-t)^{\beta-1}}{[1+x(1-t)]^{\alpha}}\,{}_2F_1\left(\begin{matrix}\alpha',\beta'\\ \gamma-\beta\end{matrix}\,\middle|\,-yt\right)dt,$$

$$R(\gamma) > R(\beta) > 0, \qquad |\arg(1+x)| < \pi, \qquad |\arg(1+y)| < \pi, \qquad (24)$$

and after the manner of proof of our previous theorems, we have the following.

THEOREM 6.

$$\left[1+\frac{\alpha(\gamma-\beta)\,x}{\gamma}\right]^{-1}\left[1+\frac{\alpha\beta\beta' y}{\gamma(\gamma-\beta)}\right]^{-1} < F_3 < \left(1-\frac{2\alpha}{1+\alpha}\right)(1-c)$$

$$+\frac{(1-(2\alpha/(1+\alpha)))\,c}{\gamma(\gamma-\beta+1)}\left[\beta+\frac{(\gamma-\beta)(\alpha+1)}{1+v'}\right]$$

$$+\frac{(2\alpha/(1+\alpha))(1-c)}{\gamma(\beta+1)}\left[\gamma-\beta+\frac{\beta(\gamma+1)}{1+v}\right]$$

$$+\frac{(2\alpha/(1+\alpha))\,c}{[fy+((1+\alpha)/2)\,x+((1+\alpha)/2)\,fxy]}$$

$$\times\left[\frac{(1+\alpha)/2)\,x}{\gamma(\beta+1)}\left\{\gamma-\beta+\frac{\beta(\gamma+1)}{1+v}\right\}\right.$$

$$\left.+\frac{fy}{\gamma(\gamma-\beta+1)}\left\{\beta+\frac{(\gamma-\beta)(\gamma+1)}{1+v'}\right\}\right]$$

$$c = \frac{(\gamma-\beta+1)\,u'}{\gamma-\beta}, \qquad f = \frac{(\alpha'+1)(\beta'+1)}{2(\gamma-\beta+1)},$$

$$0 \leqslant \alpha \leqslant 1, \quad 0 \leqslant \alpha' \leqslant 1, \quad \gamma > \beta > 0, \quad \beta' > 0, \quad x > 0, \quad y > 0, \qquad (25)$$

where u', v and v' are defined in (17).

We omit the proof. Notice that a further inequality can be derived from (25) in view of the symmetry relation (16). A straightforward calculation shows that the left side of (17) is less than the left side of (25) provided

$$\alpha x(\gamma-2\beta) < \frac{\alpha'\beta' y(\gamma-2\beta)}{\gamma-\beta}, \gamma \neq 2\beta. \qquad (26)$$

The two left side expressions are the same if $\gamma = 2\beta$.

5. Inequalities for F_4

We have

$$F_4 \equiv F_4(\alpha, \beta, \gamma, \gamma'; -x(1+y), -y(1+x)) = \frac{\Gamma(\gamma)\,\Gamma(\gamma')}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$

$$\times \int_0^1 \int_0^1 \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}\,v^{\beta-1}(1-v)^{\gamma'-\beta-1}\,du\,dv}{(1+ux)^{\gamma+\gamma'-\alpha-1}(1+vy)^{\gamma+\gamma'-\beta-1}(1+ux+vy)^{\alpha+\beta+\gamma+\gamma'-1}},$$

$$R(\gamma) > R(\alpha) > 0, \qquad R(\gamma') > R(\beta) > 0, \qquad |\arg(1+x)| < \pi,$$

$$|\arg(1+y)| < \pi, \qquad |\arg(1+x+y)| < \pi, \tag{27}$$

$$F_4 = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(-x)^m(-y)^n}{(\gamma)_m(\gamma')_n\, m!n!}, \qquad |x^{1/2}| + |y^{1/2}| < 1, \tag{28}$$

neither γ nor γ' is a negative integer or zero, and

$$F_4(\alpha, \beta, \gamma, \gamma'; -x(1+y), -y(1+x)) = F_4(\beta, \alpha, \gamma', \gamma; -y(1+x), -x(1+y)). \tag{29}$$

We can achieve an inequality for F_4 after the manner of our previous work, but a rather simple theoretical result does not emerge because the integrand of (27) contains three binomial functions. Since a more suitable integral definition of F_4 is not known, we do not further consider inequalities for F_4.

6. Inequalities for Generalized Hypergeometric Functions of an Arbitrary Number of Variables and Parameters

In the case of hypergeometric functions of a single variable, we know that a ${}_{p+1}F_{q+1}$ can be defined as a beta integral whose kernel is a ${}_pF_q$. Further, under appropriate restrictions, we can go from a ${}_pF_q$ to a ${}_{p-1}F_q$ by a confluence argument, and by the use of the Laplace transform, we can go from a ${}_pF_q$ to a ${}_{p+1}F_q$ if $p \leqslant q$ or to a certain G-function if $p = q + 1$. These same ideas carry over to hypergeometric functions of two variables.

In illustration, under appropriate restrictions on the variables and parameters which we omit, we have

$$F_1^* \equiv F_1^*(\alpha, \beta, \beta', a, b, \gamma, c, f; -x, -y) = \frac{\Gamma(c)\,\Gamma(f)}{\Gamma(a)\,\Gamma(c-a)\,\Gamma(b)\,\Gamma(f-b)}$$

$$\times \int_0^1 \int_0^1 t^{a-1}(1-t)^{c-a-1}\,u^{b-1}(1-u)^{f-b-1}\,F_1(\alpha, \beta, \beta', \gamma; -xt, -yu)\,dt\,du, \tag{30}$$

$$F_1^* = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n(a)_m(b)_n(-x)^m(-y)^n}{(\gamma)_{m+n}(c)_m(f)_n\, m!n!}. \tag{31}$$

Thus by integrating (5) with x and y replaced by xt and yu, respectively, and by applying previous techniques, we can get inequalities for $F_1{}^*$ and the process can be iterated. We pursue this no further as the details are straightforward and there seems no immediate need to have the forms for the applications.

We can also derive inequalities for hypergeometric functions of an arbitrary number of variables such as those known as Lauricella's function by starting with multiple integral representations of the Eulerian type which are obvious generalizations of the forms for F_i, $i = 1, 2, 3, 4$ (see [3]). For the most part the results are complicated because of the multiplicity of binomial functions which enter the integrands. The situation is much akin to the analysis of F_4 and we refrain from any further discussion of these generalizations at this time.

7. Extension of the Domain of Validity of the Inequalities

Recall from [1] that Kummer's transformation formulas are useful to analytically continue the ${}_2F_1$ and ${}_1F_1$ and to extend inequalities for these functions. Contiguous relations are also pertinent for such purposes. Similar type formulas are known for hypergeometric functions of two variables which the reader can find in [3]. In general, all the procedures related in [1] to extend the domain of validity of inequalities for hypergeometric functions of a single variable have their analog for hypergeometric functions of two variables. It seems that we have sufficiently elaborated on these points in this work and also in our previous study, and further commentary is unnecessary.

8. Numerical Examples

We conclude with some numerical examples.

(1) In F_1, let

$$\alpha = \tfrac{1}{2}, \qquad \beta = \tfrac{1}{4}, \qquad \beta' = \tfrac{3}{4}, \qquad \gamma = 2, \qquad x = \tfrac{1}{2}, \qquad y = \tfrac{3}{4}.$$

From (5),

$$0.85333 < F_1 = 0.87042 < 0.87241, \tag{32}$$

and from (8),

$$\frac{7383}{8687} = 0.84989 < F_1 < \frac{655687}{752675} = 0.87114. \tag{33}$$

(2) In F_3, let

$$\alpha = \tfrac{1}{4}, \quad \alpha' = \tfrac{1}{2}, \quad \beta = \tfrac{1}{3}, \quad \beta' = \tfrac{3}{4}, \quad \gamma = 2, \quad x = \tfrac{1}{2}, \quad y = \tfrac{2}{3}.$$

From (20), the remark following (20) and (21), we find

$$M(0) = 0.88617\,09062, \quad M(2) = 0.88615\,54028, \quad M(\infty) = 0.88611\,32966, \tag{34}$$

respectively. Thus

$$M(\rho) = \frac{0.88611\rho + 4.81356}{\rho + 5.43187} = 0.88611 + \frac{0.00031}{\rho + 5.54187}, \tag{35}$$

whence $M(\rho)$ is virtually independent of ρ for $\rho \geqslant 0$. Clearly $\frac{1}{4} \leqslant \rho \leqslant \frac{3}{2}$ and $M(\rho)$ is a minimum for $\rho = \frac{3}{2}$ in which event $M(\frac{3}{2}) = 0.88615$. Hence

$$0.87075 < F_3 = 0.88449 < 0.88615. \tag{36}$$

With the same basic data, from (25) we get

$$0.88357 < F_3 = 0.88449 < 0.88617. \tag{37}$$

References

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